Estimating Outgoing Quality Using the Quality Measurement Plan

Gary G. Brush and Bruce Hoadley
Bellcore
Red Bank, NJ 07701

Bernard Saperstein
Israel Aircraft Industries
Lod
Israel

Consider a sequence of lots from a manufacturing process that is put through an acceptance-sampling inspection program, the accepted lots being shipped to a customer. Taken together, the accepted lots form a population whose overall quality, called the outgoing quality, is of importance to the customer. This article describes a method for estimating this outgoing quality using data from both the accepted and rejected lots. The method is based on a very general definition of outgoing quality and on the quality measurement plan (QMP)—a hierarchical Bayes model of the underlying process. Our estimator is the only one known to us that works when sample sizes and acceptance numbers vary from lot to lot and the acceptance-sampling rules are sometimes broken—a practical reality. In the case of fixed sample size, the QMP estimator of outgoing quality has a very simple form. It is the weighted average of two naive estimators—the sample qualities observed in the incoming and outgoing lots, respectively. The weight on the observed incoming quality is monotonically increasing with [sampling variance]/[process variance].

KEY WORDS: Acceptance sampling; Bayes estimation; Empirical Bayes; Inspection.

1. INTRODUCTION

The estimation of outgoing quality is an important part of a quality-assurance program. Stated simply, the outgoing quality is the final quality level (nonconformances per unit) shipped to the customer. The concept of “average outgoing quality” was used by Dodge and Romig (1959) to develop the family of average outgoing quality limit sampling plans.

Outgoing quality is usually estimated by an independent quality audit of product that has been declared ready for shipment by the manufacturing organization. Hoadley (1981) described such an audit and provided an estimator of outgoing quality based on the quality measurement plan (QMP). Sometimes, however, the only quality data available are from a final inspection process in which samples are taken and some lots (or portions of lots) are rejected and some accepted. The problem of estimating outgoing quality from such an acceptance-sampling plan is not as simple as for an audit. Hahn (1986) provided an introductory discussion of this problem and also proposed a very simple estimator for special cases.

Many companies use the nonconformances per unit in the samples from accepted lots as an estimator of outgoing quality. This is clearly optimistic because, for example, if the acceptance number is 0, this naïve estimator is zero nonconformance per unit no matter how many lots are rejected. Others have advocated using the nonconformances per unit in all of the samples taken. This is clearly pessimistic because the underlying manufacturing process may be out of statistical control and the rejection process may be very effective in improving quality.

There is some literature on characterizing how acceptance sampling improves quality. Brush, Cautin, and Lewin (1981) showed how the distribution of outgoing quality varies as a function of the incoming quality distribution when MIL-STD-105D (U.S. Department of Defense 1963) sampling plans are used. Liebesman (1979) developed the concept of average outgoing quality maximum. This is the limiting outgoing quality which results from a sampling system in which quality improves when tightened inspection is used.

The problem has received attention lately because suppliers are being required to provide data to demonstrate that their products are meeting specified quality standards. There is little literature on how to actually estimate outgoing quality from an acceptance-sampling plan, however. Hoadley and Saperstein (1978) proposed empirical Bayes estimators under several assumptions about the incoming process distribution. Zaslavsky (1988) derived an estimator based on a nonparametric empirical Bayes argument.
All of these estimators suffer from various drawbacks. For example, Hahn’s (1986) estimator requires that the lot size, sample size, and acceptance number remain constant from lot to lot. The estimators of both Hahn (1986) and Zaslavsky (1988) require (a) that the sample size be much smaller than the lot size and (b) that the acceptance-sampling plan follow the rules—an unlikely event because when the situation is desperate, bad lots are shipped. In addition, they provide no interval estimator for the case of varying sample sizes. The Hoadley–Saperstein (1978) estimators have associated posterior variances, but they are based on a so-called “naive empirical Bayes” approach in which they ignore the uncertainty introduced by the estimation of the incoming process distribution.

In this article, we show that these problems can be solved by combining the ideas of Hoadley and Saperstein (1978) with the hierarchical Bayes approach of Hoadley’s (1981, 1984) QMP and Bellcore (1987). In the case of fixed sample size, the QMP estimator of outgoing quality has a very simple and intuitive form. It is the weighted average of two naive estimators of outgoing quality: \( \hat{\omega} \cdot \hat{OQ}_1 + (1 - \hat{\omega}) \cdot \hat{OQ}_2 \), where \( \hat{OQ}_1 \) and \( \hat{OQ}_2 \) are the sample nonconformances per unit observed in the incoming and outgoing lots, respectively. The weight has the form:

\[
\hat{\omega} = \frac{[\text{sampling variance}]}{[\text{sampling variance}] + [\text{process variance}]}.
\]

So if the process is very stable (process variance \( \approx 0 \)), then the QMP estimator is just the observed incoming quality, but if the sample size is very large (sampling variance \( \approx 0 \)) and/or the process is very unstable (process variance large), then the QMP estimator is just the quality observed in the accepted lots. Otherwise, the QMP estimator is in between. For QMP, \( \hat{\omega} \) is adaptively estimated from the data.

The QMP idea can also be used when one is interested in the quality of the current accepted lot. The QMP estimator is then just \( \hat{\omega} \cdot \hat{OQ}_1 + (1 - \hat{\omega})(x/n) \), where \( x/n \) is the observed nonconformances per unit in the current accepted lot.

The article starts out by defining incoming and outgoing quality. Then we discuss the pros and cons of various estimators that were discussed previously. We then show how to estimate outgoing quality using QMP. Finally, we discuss a few examples for illustration and then use simulation to study behavior in a variety of settings.

The article shows that QMP provides a very general and useful method for estimating (point and interval) outgoing quality. QMP has an excellent percentage of error and interval-estimate coverage properties for many cases that might arise in practice. QMP is not universally robust to deviation from the assumed unknown gamma process distribution, however. Deviations to watch out for are extreme bimodality and very thick tails (i.e., outliers). Moreover, QMP constructs a Bayes estimate of the process distribution. This requires a prior distribution on the so-called hyperparameters (process average, process variance). Users should select this prior distribution carefully and empirically for the collection of products under consideration.

QMP theory is based on a Poisson assumption for the number of nonconformances in the sample. For some inspections, however, the unit is either good or bad. Then the number of nonconformances in the sample is binomial. Section 5.4 provides a discussion of how to handle this case. The article also shows that the commonly used naive estimators generally perform poorly and should not be used.

2. PROBLEM DEFINITION AND SAMPLING DISTRIBUTION

The observed variables for lot \( i \) are \( N_i = \text{lot size} \) \((i = 1, \ldots, T)\) (data window size), \( n_i = \text{sample size} \), \( c_i = \text{acceptance number} \), \( x_i = \text{number of nonconformances in the sample} \), \( M_i = \text{number of items accepted} \), and \( M_{ui} = \text{number of uninspected items accepted} \). We define \( x = (x_1, \ldots, x_T) \).

Some unobserved variables of interest for lot \( i \) are \( Y_i = \text{number of nonconformances accepted} \), \( X_i = \text{number of nonconformances in the lot at the time of submittal} \), and \( Z_i = \text{number of nonconformances in the unsampled portion of the lot} \).

We have observed in practice that accepting and rejecting lots is not always a yes-or-no proposition. For example, when the lot sizes are small and the sample is thus a sizable portion of the lot, the “good” units in the sample might be accepted, but the “bad” units in the sample and the uninspected units are rejected. Another example is the case in which the customer is desperate for product and the lot is accepted even though \( x_i > c_i \). Or maybe one-half of such a lot is accepted. All of these unusual cases can be handled by our method as long as two reasonable assumptions are made: (a) when the uninspected portion of the lot is rejected, the nonconforming units in the sample are rejected or fixed and (b) no nonconforming unit in the sample is accepted without being fixed. Under these assumptions, it is true that if \( M_{ui} > 0 \), then \( Y_i = Z_i \), and if \( M_{ui} = 0 \), then \( Y_i = 0 \). These two facts will be used later. Note also that our method requires one to keep track of \( M_i \) and \( M_{ui} \).

For reporting purposes, suppose that we are interested in a subset of the lots, labeled for convenience \( 1, \ldots, k \). Call \( K \) the reporting-window size. This subset might represent, for example, a partic-
usual month. In many applications, \( K = T \). Define

\[
\text{IQ} = \sum_{i=1}^{K} X_i / \sum_{i=1}^{K} N_i, \quad \text{OQ} = \sum_{i=1}^{K} Y_i / \sum_{i=1}^{K} M_i, 
\]

which will be referred to as incoming quality (IQ) and outgoing quality (OQ), respectively. The main thrust of this article is to estimate OQ.

Our sampling distribution model is:

1. For lot \( i \), there exists an unknown true nonconformance per unit rate, \( \lambda_i \).
2. The production process produces a lot of size \( N_i \) with \( X_i \) nonconformances. We assume

\[
X_i \mid \lambda_i \sim \text{Poisson} (N_i \lambda_i). 
\]

3. The acceptance-sampling inspection selects a random sample of size \( n_i \), without replacement from the lot and observes \( x_i \) nonconformances. So, conditional on \( \lambda = (\lambda_1, \ldots, \lambda_T) \), the variables \( x_1, \ldots, x_T, Z_1, \ldots, Z_T \) are statistically independent with \( x_i \mid \lambda_i \sim \text{Poisson} (n_i \lambda_i) \) and \( Z_i \mid \lambda_i \sim \text{Poisson} ((N_i - n_i) \lambda_i) \) \( (i = 1, \ldots, T) \).

3. ESTIMATORS OF OUTGOING QUALITY IN THE LITERATURE

First we note that there are several naive estimators of outgoing quality that can be badly biased. The first naive estimator uses the observed incoming quality based on all of the data to estimate \( Y_i \) by

\[
\hat{Y}_i = M_{ni} \left[ \sum_{j=1}^{K} X_j / \sum_{j=1}^{K} n_j \right]
\]

so that the estimator of outgoing quality is

\[
\hat{OQ}_1 = \sum_{i=1}^{K} \hat{Y}_i / \sum_{i=1}^{K} M_i. 
\]

This ignores the fact that acceptance sampling may improve quality. For example, suppose that the manufacturing process produces only two types of lots, those with no nonconforming units and those with only nonconforming units. In that case \( OQ = 0 \), but \( \hat{OQ}_1 > 0 \).

The second naive estimator uses only data from the accepted lot to estimate \( Y_i \) by \( \hat{Y}_i = M_{ni}(x_i/n_i) \), so the estimator of outgoing quality is

\[
\hat{OQ}_2 = \sum_{i=1}^{K} \hat{Y}_i / \sum_{i=1}^{K} M_i. 
\]

Now suppose that \( c_i = 0 \) for all \( i \) and \( X_i/n_i = p \). Then \( OQ = p \), but \( \hat{OQ}_2 = 0 \). \( \hat{OQ}_2 \) ignores the process relationship of lots.

In the case in which \( N_i = N \), \( n_i = n \), \( c_i = c \), and \( n \ll N \), Hahn (1986) and Zaslavsky (1988) (Zaslavsky also treats unequal sample sizes) have proposed estimators based on \( T_i = \) number of lots with \( x_i = x \), where \( \sum_{i=0}^{c} T_i = K \). The Zaslavsky (1988) estimator is

\[
\hat{OQ}_Z = \left( \frac{1}{n} \right) \frac{\sum_{i=0}^{c} x T_i}{\sum_{i=0}^{c} T_i}. 
\]

Zaslavsky showed that if \( \lambda_1, \ldots, \lambda_T \) is a random sample from an unknown process distribution, \( \pi \), then

\[
E[\lambda_i \mid x_i \leq c, \pi] = \left( \frac{1}{n} \right) \frac{\sum_{i=0}^{c} x T_i}{\Pr(x_i \leq c \mid \pi)} 
\]

To estimate this, just estimate \( \Pr(x_i = x \mid \pi) \) by \( T_i/K \). Fortunately, \( \pi \) itself does not have to be estimated. This is a nonparametric empirical Bayes approach. For a general discussion of empirical Bayes, see Maritz and Lwin (1989).

For this special case, where we assume that \( M_{ni} = M \), and

\[
M_i = N \quad \text{if } x_i \leq c \\
0 \quad \text{if } x_i > c.
\]

we can write the naive estimator \( \hat{OQ}_2 \) as

\[
\hat{OQ}_2 = \left( \frac{1}{n} \right) \frac{\sum_{i=1}^{c} x T_i}{\sum_{i=1}^{c} T_i}.
\]

The only difference between \( \hat{OQ}_Z \) and \( \hat{OQ}_2 \) is that \( \hat{OQ}_Z \) has the extra term \( (c + 1) T_{c+1} \) in the numerator. This points out a potential drawback of \( \hat{OQ}_Z \). Suppose that \( c = 0 \), and out of 10 lots, \( T_0 = 5, T_1 = 0, T_2 = 2, T_3 = 2, \) and \( T_4 = 1 \). In this case \( \hat{OQ}_Z \) = 0, but the fact that 50% of the lots were rejected casts doubt on the quality of the accepted lots. The estimator is completely dependent on \( T \) to carry the information on possible nonconformances in the accepted lots, and \( T \) can have a large variance, particularly when there are few lots.

Zaslavsky (1988) also discussed a parametric empirical Bayes estimator based on Formula (6). Maritz and Lwin (1989) and Hoadley and Saperstein (1978) also proposed such estimators. The idea is to assume that \( \pi \) is a gamma distribution with unknown mean and variance, which are estimated using a maximum likelihood estimator or the method of moments. Zaslavsky compared these estimators with (5) based on asymptotic relative efficiency. This approach seems simple enough, but in our extensive experience, it does not always work. For example, if \( x_i = 0 \) for \( i = 1, \ldots, T \) (a real case), everything degenerates to 0, even the interval estimate. This was not acceptable to our clients, who wanted a plausible mea-
sure of the uncertainty of the estimate, so we do not consider this approach further.

The QMP approach amounts to using a parametric Bayes estimator of $\pi$. The formulas are more complex but are still algebraic. These QMP formulas arose out of a lengthy fine-tuning process that lasted for the two-year trial of QMP at Western Electric. During that time, QMP was challenged by 20,000 real data sets (some very strange) and a lot of skeptical quality engineers. QMP had to pass a sanity test for every case. Our prior used in estimating $\pi$ was empirically derived using data across hundreds of products—each with its own $\pi$. This injection of additional information acts as an enforcer to keep the results sane. After all, attributes data are not always all that informative.

From this discussion, it is clear that available estimators of outgoing quality all have limitations. We seek an estimator with the following characteristics:

1. The estimator can handle cases in which $N_i$, $n_i$, and $c$ vary arbitrarily from lot to lot.
2. The estimator can handle cases in which the acceptance-sampling rules are violated; for example, a lot with $x_i > c_i$ is 50% accepted.
3. The estimator has good percentage-of-error properties for a wide variety of scenarios.
4. An interval estimator can be computed.

In Sections 4 and 5, we derive such an estimator based on the concepts of Hoadley (1981) and Hoadley and Saperstein (1978).

4. A BAYES ESTIMATOR OF OUTGOING QUALITY

Our sampling distribution was defined in Section 2. With the Poisson distribution, we are explicitly treating nonconformances (defects) rather than nonconforming units (defectives). See Section 5.4 for a discussion of defectives. Furthermore, we assume that $\lambda$ has a known prior distribution. Later in the article, we will propose a specific hierarchical prior for $\lambda$, but for now we explore the more general setting.

From Equation (1), it follows that the Bayes estimator of outgoing quality (along with its posterior variance) is

$$E(OQ | x) = \frac{\sum_{i=1}^{K} E(Y_i | x)}{\sum_{i=1}^{K} M_i}$$  (7)

and

$$V(OQ | x) = \left[ \frac{1}{\sum_{i=1}^{K} M_i} \right]^2 \times \left[ \sum_{i=1}^{K} V(Y_i | x) + 2 \sum_{j=2}^{K} \sum_{i=1}^{K} C(Y_j, Y_i | x) \right]$$  (8)

where $C(Y_i, Y_j | x)$ denotes the conditional covariance. To use this, we need to determine $E(Y_i | x)$, $V(Y_i | x)$, and $C(Y_i, Y_j | x)$.

4.1 Posterior Mean of $Y_i$

If $M_{ui} > 0$, then $Y_i = Z_i$, so

$$E(Y_i | x) = E(Z_i | x) = E(E(Z_i | \lambda_i, x) | x) = E(M_{ui} \lambda_i | x) + M_{ui}^2 V(\lambda_i | x) = M_{ui} E(\lambda_i | x).$$  (9)

If $M_{ui} = 0$, then $Y_i = 0$ and (9) still holds.

4.2 Posterior Variance of $Y_i$

If $M_{ui} > 0$, then $Y_i = Z_i$, and

$$V(Y_i | x) = V(Z_i | x) = E(V(Z_i | \lambda_i, x) | x) + V(E(Z_i | \lambda_i, x) | x) = E(M_{ui} \lambda_i | x)^2 V(\lambda_i | x) = M_{ui} E(\lambda_i | x) + M_{ui}^2 V(\lambda_i | x).$$  (10)

If $M_{ui} = 0$, then $Y_i = 0$ and $V(Y_i | x) = 0$. But since $M_{ui} = 0$, (10) holds here as well.

4.3 Posterior Covariance of $Y_i$ and $Y_j$

If both $M_{ui} > 0$ and $M_{uj} > 0$, then $Y_i = Z_i$ and $Y_j = Z_j$, and

$$C(Y_i, Y_j | x) = E(C(Y_i, Y_j | \lambda_i, \lambda_j, x) | x) + C(E(Y_i | \lambda_i, \lambda_j, x), E(Y_j | \lambda_i, \lambda_j, x) | x) = E(C(Z_i, Z_j | \lambda_i, \lambda_j, x), E(Z_i | \lambda_i, \lambda_j, x) | x) + C(E(Z_i | \lambda_i, \lambda_j, x), E(Z_j | \lambda_i, \lambda_j, x) | x) = M_{ui} M_{uj} C(\lambda_i, \lambda_j | x).$$  (11)

Note that if $M_{ui} = 0$, then $Y_i = 0$, and if $M_{uj} = 0$, then $Y_j = 0$. In either case $C(Y_i, Y_j | x) = 0$ and (11) holds.

Now we can write the general solution:

$$E(OQ | x) = \frac{\sum_{i=1}^{K} M_{ui} E(\lambda_i | x)}{\sum_{i=1}^{K} M_i}$$  (12)

and

$$V(OQ | x) = \left\{ \sum_{i=1}^{K} \left[ M_{ui} E(\lambda_i | x) + M_{ui}^2 V(\lambda_i | x) \right] \right\} + 2 \sum_{j=2}^{K} \sum_{i=1}^{K} M_{ui} M_{uj} C(\lambda_i, \lambda_j | x) \left/ \left[ \sum_{i=1}^{K} M_i \right]^2 \right. \right.$$  (13)

This is as far as we can go without defining a specific prior distribution for $\lambda$.\"
5. APPLICATION OF THE QUALITY MEASUREMENT PLAN

5.1 The QMP Model

From Equations (12) and (13), we see that three expressions are needed—\(E(\lambda_i \mid x)\), \(V(\lambda_i \mid x)\), and \\(C(\lambda_i, \lambda_j \mid x)\) for \(i \neq j\). To obtain these expressions, we propose using the QMP (Bellcore 1987; Hoadley 1981, 1984).

QMP was implemented throughout Western Electric in 1980 and by Bellcore in 1984. It is a hierarchical Bayes approach to the control chart. Most control charts are just running hypothesis tests. QMP provides connected box-and-whisker plots or "I" plots for posterior distributions of quality as a function of time.

To apply QMP, we first make a transformation to a common index scale. Let \(s_i\) = the standard nonconformance rate for lot \(i\) (e.g., the average quality limit). Define \(\theta_i = \lambda_i/s_i\) as the true quality parameter on an index scale. So if \(\theta_i = 1\), then the true nonconformance rate \(\lambda_i\) is equal to the standard nonconformance rate \(s_i\); if \(\theta_i = 2\), then the true nonconformance rate is twice as big as the standard rate.

With this transformation to an index scale, it is reasonable to assume that \(\theta_1, \ldots, \theta_T\) are a priori exchangeable. This suggests the following hierarchical Bayes model. Conditional on the process average \(\theta\) and the process variance \(\gamma^2\), \(\theta_1, \ldots, \theta_T\) is a random sample from a gamma distribution with unknown mean \(\theta\) and variance \(\gamma^2\). In hierarchical Bayes parlance, \(\theta\) and \(\gamma^2\) are called hyperparameters. Finally, we assume that \((\theta, \gamma^2)\) has a known prior distribution.

We never specify the exact form of this prior—only some characteristics like means, variances, and percentiles (see Sec. 5.2). These prior parameters are estimated from data across many products. This is made possible by the index scale. The gamma distribution is used because it is the natural conjugate prior to the Poisson and it is a reasonable parametric model of a unimodal distribution on the nonnegative real numbers.

For this model, heuristic algorithms were developed by Hoadley (1981, 1984) and Bellcore (1987) for \(E(\theta \mid x), V(\theta \mid x), \) and \(C(\theta_i, \theta_j \mid x)\) (\(i \neq j\)). Note that the expressions needed for this article are

\[
\begin{align*}
E(\lambda_i \mid x) &= s_i E(\theta_i \mid x) \\
V(\lambda_i \mid x) &= s_i^2 V(\theta_i \mid x) \\
C(\lambda_i, \lambda_j \mid x) &= s_i s_j C(\theta_i, \theta_j \mid x), \quad i \neq j. \tag{14}
\end{align*}
\]

5.2 The QMP Formulas

Two heuristic algorithms have been implemented for QMP. The first was derived by Hoadley (1981) and the second, which is currently used at Bellcore, was derived by Hoadley (1984). For completeness, we state the formulas from Hoadley (1984) that are needed to implement the QMP estimator of outgoing quality. The formulas look complex, but they are algebraic and easily programmable. Notice that we use the data from all lots \(1, \ldots, T\) to estimate \(OQ\) for the subset \(1, \ldots, K\). For \(i = 1, \ldots, T\), let

\[
e_i = \text{expected number of nonconformances in the sample} \]

\[
= n_s \]

and

\[I_i = \text{sample quality index} \]

\[= x_i/e_i.\]

These are the input observations to the QMP algorithm.

The formulas for the required posterior moments of \(\theta_1, \ldots, \theta_T\) are

\[
E(\theta \mid x) = \hat{\omega}_0 \hat{\theta} + (1 - \hat{\omega}_0) I_i = \hat{\theta}, \tag{15}
\]

\[
V(\theta \mid x) = (1 - \hat{\omega}_0) \hat{\theta}/e_i + \hat{\omega}_i \sum_{i=0}^r \frac{\hat{\gamma}^2 + \hat{\theta}^2}{e_i} + \frac{r_i^2(\hat{\theta} - I_i)^2}{(r_i - 1)\hat{\omega} + 1} G, \tag{16}
\]

and

\[
C(\theta_1, \theta_2 \mid x) = \hat{\omega}_0 \hat{\omega}_1 \sum_{i=0}^r \frac{\hat{\gamma}^2 + \hat{\theta}^2}{e_i} + \frac{r_i}{(r_i - 1)\hat{\omega} + 1} \times \left[\frac{r_i}{(r_i - 1)\hat{\omega} + 1}\right] G, \tag{17}
\]

where expressions for \(\hat{\omega}_0, \hat{\theta}, p_i, \hat{\gamma}^2, r_i, G, \hat{\omega}, \) and \(e_0\) are as will be stated subsequently.

We see that \(E(\theta \mid x)\) is just a weighted average of the process average \(\hat{\theta}\) and the observed index \(I_i\) in lot \(i\). It shrinks \(I_i\) toward the process average. The variable \(\hat{\omega}_0\) is called the shrinkage weight and is estimated adaptively. It is an estimate of \(\omega = [\theta/e_i + \hat{\gamma}^2]\), and \(\theta/e_i = E(V(I_i \mid \lambda_i))\), which is the expected sampling variance. So \(\omega\) is of the generic form

\[
\frac{\text{sampling variance}}{\text{sampling variance} + \text{process variance}}.
\]

The expression for \(V(\theta \mid x)\) has three components. The first component is the simple posterior variance if the process average \(\hat{\theta}\) and the process variance
As explained in Section 5.1, the QMP model has a prior distribution on \( (\theta, \gamma^2) \). The parameters of this prior that we use are \( \theta_0 = \) prior mean of \( \theta \), \( v_0 = \) prior variance of \( \theta \), \( \gamma_0^2 = \) prior mean of \( \gamma^2 \), and \( \gamma_{\text{max}}^2 \) is defined by \( \text{Pr}(\gamma^2 \leq \gamma_{\text{max}}^2) = .95 \). We require a technical constraint \( v_0 > \gamma_0^2 \), which is caused by some of the prior information being implemented in the algorithm as artificial data:

\[
e_0 = \text{prior expectancy} = \theta_0 / (v_0 - \gamma_0^2)
\]

and

\[
x_0 = \text{prior nonconformances} = \theta_0 e_0.
\]

See Hoadley (1984, p. 27) for a detailed explanation.

The default priors used in the current Bellcore implementation of QMP are \( \theta_0 = 1 \), \( v_0 = 3.05 \), \( \gamma_0^2 = .55 \), and \( \gamma_{\text{max}}^2 = 2.2 \). These were arrived at by an analysis of factory quality-audit data for many products. We also considered the ranges of \( \theta \) and \( \gamma^2 \) for which good mean squared error properties were important. The default prior is used in Sections 6.2 and 7.

To test the QMP algorithm on the actual data in this article, we also use a diffuse prior on \( (\theta, \gamma^2) \). The diffuse prior selected was \( \theta_0 = 5 \), \( v_0 = 240 \), \( \gamma_0^2 = 200 \), and \( \gamma_{\text{max}}^2 = 760 \).

The next step in the algorithm is to compute certain weights that are used to accumulate data over lots. Weights are needed because there is lot-to-lot variation in sample size and thus expectancy. For \( t = 0, 1, \ldots, T \), let \( b_t = \gamma_0^2 + \theta_0 / e_t \), \( f_t = 1 / b_t \), and \( g_t = 1 / (\phi_0 b_t^2 - 4 \gamma_0^2 b_t / (\theta_0 e_t)) \), where \( \phi_0 = 2 + 6 \gamma_0^2 / \theta_0^2 \). The required weights for \( t = 0, 1, \ldots, T \) are

\[
p_t = f_t / \sum_{i=0}^{T} f_i,
q_t = g_t / \sum_{i=0}^{T} g_i.
\]

To compute the variables needed in Formulas (15)–(17), we need some preliminary expressions:

1. Process average: \( \hat{\theta} = \sum_{t=0}^{T} p_t I_t \).
2. Average sampling variance: \( \sigma^2 = \sum_{t=1}^{T} q_t (\hat{\theta} / e_t) \).
3. Total observed variance: \( V = \sum_{t=1}^{T} q_t (I_t - \hat{\theta})^2 \).
4. Shape parameter for the sampling distribution of \( V \):

\[
a_1 = \frac{\left( \sum_{t=1}^{T} q_t (\hat{\theta} / e_t) \right)^2}{\sum_{t=1}^{T} q_t [2(\hat{\theta} / e_t)^2 + \theta_0 / e_t^2]}.
\]

5. Shape parameter for the prior distribution of \( \omega = \sigma^2 / (\sigma^2 + \gamma^2) \):

\[
a_0 = \frac{\ln(20)}{\ln[1 + \gamma_{\text{max}}^2 / \sigma^2]}.
\]

6. Shape parameter of the posterior distribution of \( \omega: a = a_0 + a_i \).
7. Adjusted total variance: \( S^2 = (a_i / a) V \).
8. Adjusted variance ratio: \( R = S^2 / \sigma^2 \).
9. Bayes adjustment factor used to keep the estimated process variance positive: \( F = 1 + (1 / \Sigma_{11}) \) (for \( R > 0 \)), where \( \Sigma_{11} = \Sigma_{n=1}^{m} T(i) \), where \( T(0) = 1, T(i) = T(i - 1) [a R / (a + i)] \), and \( T(m) \) is the term in which either \( T(m) > 10^7 \) or \( T(m) < 10^{-7} \).
10. Posterior variance of \( \omega \):

\[
G = \left[ \frac{(a + 1)}{a R} - F + 1 - \frac{1}{FR} \right] / FR \quad \text{if} \ R > 0
\]

\[
= \frac{a}{(a + 2)(a + 1)^2} \quad \text{if} \ R = 0.
\]

We can now compute the variables needed for Formulas (15)–(17):

1. Sampling variance for lot \( i \): \( \sigma_i^2 = \hat{\theta} / e_t \).
2. Sampling variance ratio: \( r_i = \sigma_i^2 / \sigma^2 \).
3. Process variance:

\[
\hat{\sigma}^2 = FS^2 - \sigma^2 \quad \text{if} \ R > 0
\]

\[
= \sigma^2 / a \quad \text{if} \ R = 0.
\]
4. Average shrinkage weight: \( \hat{\omega} = \sigma^2 / (\sigma_i^2 + \hat{\sigma}^2) \).
5. Shrinkage weight for lot \( i \): \( \omega_i = \sigma_i^2 / (\sigma_i^2 + \hat{\sigma}^2) \).

5.3 A Simple Form for the QMP Estimator

Consider the simple case in which \( K = T \), the sample sizes are equal \( (n, = n) \), and the standards are equal \( (s_i = s) \); consequently, the weights are equal \( (\omega_i = \omega) \). In this case

\[
E(\lambda_i | x) = sE(\theta_i | x)
\]

\[
= \hat{\omega} s \hat{\theta} + (1 - \hat{\omega})(x_i / n_i);
\]

and from (12)

\[
E(OQ | x) = \hat{\omega} \left[ \frac{\sum_{i=1}^{K} M_i (s \hat{\theta})}{\sum_{i=1}^{K} M_i} \right] + (1 - \hat{\omega}) \left[ \frac{\sum_{i=1}^{K} M_i (x_i / n_i)}{\sum_{i=1}^{K} M_i} \right].
\]
Now $s\hat{\theta}$ is just the Bayes estimator of the incoming process average for nonconformance per unit and is approximately $\sum_{i=1}^{r} x_i/\sum_{i=1}^{r} n_i$. So by (3) and (4),

$$E(OQ \mid x) = \hat{\omega} \cdot \hat{O}_Q + (1 - \hat{\omega}) \cdot \hat{O}_Q^2. \quad (18)$$

This is a simple intuitive estimator. The weight $\hat{\omega}$ is of the form described in the introduction.

5.4 The Binomial Case

Suppose that for lot $i$, you only observe $x_i = \text{[number of nonconforming units in the sample]}$ and $\lambda_i = \text{[unknown but true fraction of nonconforming units]}$. Then $x_i \mid \lambda_i \sim \text{binomial}(n_i, \lambda_i)$. There are two QMP-type approaches to this problem. One is to just approximate binomial $P(n_i, \lambda_i)$ to Poisson $(n_i, \lambda_i)$. This works if $\lambda_i (1 - \lambda_i) = \hat{\lambda}$. Another approach would be to first derive formulas analogous to (12) and (13) for the binomial case. Formula (14) would still hold with $s_i = \text{[standard fraction of nonconforming units]}$. This reduces the problem to computing $E(\theta_i \mid x)$, $V(\theta_i \mid x)$, and $C(\theta_i, \theta_i \mid x)$.

To compute these posterior moments, a beta/bi-nomial version of QMP could be derived. To avoid that work, QMP could be applied using the concept of equivalent defects and expectancy explained by Hoadley and Saperstein (1978). However. Such an approach would have a large effect on the process variance, so the QMP user should attend to the outliers.

A very interesting aspect of this is that the QMP estimate increases when the outliers are removed. This is counter-intuitive. The outliers provide further evidence (recognized by QMP) of greater process variance. Greater process variance implies easier discrimination between good and bad lots. This implies that outgoing quality will be better.

In Example 1, $\hat{O}_Q \approx \hat{O}_Q$. This is because $T_s = 0$ and $\hat{O}_Q$ ignores the lot with four nonconformances. The QMP estimate is closer to $\hat{O}_Q$; however, because the sample provides evidence that the underlying process is in reasonable statistical control around .0027 nonconformances per unit, which is near the standard of .0025. This suggests that there are plenty of nonconformances in the accepted lots and the acceptance sampling is not improving quality very much. The simulation in Section 7 will shed more light on this issue.

The 90% interval estimate in Table 3 is computed from the 5th and 95th percentiles of a gamma distribution, which is fitted by the method of moments using $E(OQ \mid x)$ and $V(OQ \mid x)$. In Example 2b, the Zaslavsky estimate is not even contained in the QMP 90% interval estimate. So the question arises—where is right? Of course, with real data you cannot tell. But the question is addressed in Section 8, where we describe a special-purpose simulation to study this example.

6. EXAMPLES

Some real-data examples are described in Tables 1 and 2. For simplicity, we assume that (a) $K = T$; (b) when $x_i > c_i$, the entire lot is rejected; and (c) when $x_i \leq c_i$, nonconforming units in the sample are fixed.

These examples are actual data from the electronics industry. Example 1 is taken from Hoadley and Saperstein (1978). Example 2 is taken from Hahn (1986). He did not specify the standard for Example 2, so we assumed $s = .001$, which would make the plan consistent with MIL-STD-105D (U.S. Department of Defense 1963). Example 2b is just Example 2a with the two outliers deleted.

The estimates of outgoing quality for the examples are given in Table 3. For QMP, we used the diffuse prior described in Section 5.2.

The key to understanding the QMP estimator is the shrinkage weight $\hat{\omega} = \sigma^2 / (\sigma^2 + \gamma^2)$. Table 4 lists $\gamma^2$, $\sigma^2$, and $\hat{\omega}$ for the examples.

6.1 Interpretations

For Example 1, the QMP shrinkage weight is .63, so $E(OQ \mid x)$ is closer to $\hat{O}_Q$ than $\hat{O}_Q^2$. The fact that it is so close to $\hat{O}_Q$ seems to violate Equation (18). But Equation (18) is only an approximation. The QMP estimator is really a shrinkage towards a Bayes estimator of the process average, which in this example is .0032.

For Examples 2a and 2b, the QMP estimate puts more of the weight on $\hat{O}_Q$. This is because the process variance is large and consequently the shrinkage weight is small. In Example 2a, there are two outlier lots. These outliers have a large effect on the process variance, so the QMP user should attend to the outliers.

6.2 Sensitivity of the QMP Estimator to the Prior

As discussed in Section 5.2, QMP uses a prior distribution on the hyperparameters $(\theta, \gamma^2)$. There we defined two priors called "default" and "diffuse."
The sensitivity of the QMP estimator to the prior is illustrated in Table 5.

For Example 2b, the prior makes a big difference. The difference is caused by very different estimates of process variance. The default prior is very informative. And even though there are 303 lots, the expectancy per lot is only \( e = .125 \), so there is not a huge amount of data available to estimate process variance.

This means that the selection of the prior on \( \theta \), \( \gamma^2 \) should be done with great care. We have used different priors for different applications. In a particular system application, there are usually past data available on many products. For example, our default prior was estimated from a factory audit system in which the audit was preceded by an adequate statistical process-control system. In a field-installation audit application, we used a more diffuse prior. This is because there was no statistical process control and the past audit data showed many installation vendors out of control.

In summary, these examples show that (a) the estimators are quite different, (b) the Zaslavsky (1988) estimate can fall outside of the QMP 90\% interval estimate, (c) the QMP estimator is sensitive to outliers and the Zaslavsky estimator is not, and (d) the QMP estimate can be sensitive to the prior distribution on the hyperparameter (process average, process variance). To get a better understanding of how to deal with all this, we use simulation.

7. SIMULATION STUDY

The QMP estimator is Bayesian and, therefore, minimizes average mean squared error for the model assumed. It is also the only estimator known to us that allows arbitrary lot-to-lot variation in \( N_i, n_i, \) and \( c_i \) and does not require that any particular acceptance rule be used, so if the assumed model is reasonable for a particular application, then one could safely use the QMP estimator. It is of interest, however, to compare the performance of the QMP estimator with other estimators under a variety of scenarios.

Brush, Hall, and Huston (1985) studied several estimators of OQ under a variety of process distributions using simulation. The competitors to QMP considered were (a) \( \hat{O}_1 \), (b) \( \hat{O}_2 \), (c) an estimator derived under the assumption that rejected lots are reworked with an assumed efficiency and then shipped, and (d) a QMP-type estimator derived for the skip-lot case. The findings of the study were that the QMP-based estimator, \( E(\hat{O(Q|x)} \), performed best in most cases.

At the time of this simulation, the estimator of Zaslavsky (1988) was not known to us, so it is of interest to compare QMP with this estimator.

7.1 Simulation Design

Our simulation design is the following:

1. The reporting-window size is equal to the data-window size; that is, \( T = K \). We use two values, 10 and 50. For short, we call this the window size.
2. \( N_i = N, n_i = n, c_i = c, \) and \( s_i = s \).
3. If \( x_i > c_i \), the entire lot is rejected, and if \( x_i \leq c_i \), the nonconforming units in the sample are fixed.
4. The sampling distribution in Section 2 is used.
5. \( \theta_1, \ldots, \theta_T \) is a random sample from a process distribution. The six used are (a) point mass at 1 (i.e., \( \theta_1 = 1 \)); (b) point mass at 1.5 (i.e., \( \theta_1 = 1.5 \)); (c) gamma distribution with mean = 1, variance = 1; (d) gamma distribution with mean = 2, variance = 2; (e) two-point distribution with \( \theta_1 = 1 \) or 2 and \( \Pr(\theta_1 = 1) = .75 \); and (f) two-point distribution with \( \theta_1 = .5 \) or 4 and \( \Pr(\theta_1 = .5) = .5 \).
6. The three lot/sample scenarios used are (a) \( N = 200, n = 32, s = .005, c = 0 \); (b) \( N = 500, n = 50, s = .015, c = 1 \); and (c) \( N = 1,000, n = 125, s = .025, c = 7 \).

Table 3. Estimates of Outgoing Quality for Examples

| Ex. | \( \hat{\hat{O}} \) | \( \hat{\hat{O}} \) | \( \hat{\hat{O}} \) | \( E(\hat{O(Q|x)} \) | 90\% interval estimate |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | .0027           | .0017           | .0018           | .0026           | [.0013, .0044]  |
| 2a  | .0085           | .00              | .0017           | .0002           | [.00007, .0004] |
| 2b  | .0060           | .00              | .0017           | .0009           | [.0005, .0013]  |

Table 4. QMP Statistics

<table>
<thead>
<tr>
<th>Ex.</th>
<th>Process variance ( \sigma^2 )</th>
<th>Sampling variance ( \sigma^2 )</th>
<th>Shrinkage weight ( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.51</td>
<td>2.58</td>
<td>.63</td>
</tr>
<tr>
<td>2a</td>
<td>2.612</td>
<td>70</td>
<td>.03</td>
</tr>
<tr>
<td>2b</td>
<td>191</td>
<td>41</td>
<td>.18</td>
</tr>
</tbody>
</table>

Table 2. Nonconformance Data for Examples

| Example | \( x \) | 0 | 1 | 2 | 3 | 4 | 5 | 8 | 10 | 11 | 14 | 15 | 34 | 106 | No. of lots |
|---------|--------|---|---|---|---|---|---|---|---|---|----|----|----|-----|-------|----------|
| 1       |        | 11| 6 | 0 | 0 | 1 |   |   |   |   |    |    |    |     | 18     |
| 2a      |        | T,| 227| 48 |10 | 4 | 5 | 1 | 3 | 1 | 2 | 1 | 1 | 1    | 305    |
| 2b      |        | T,| 227| 48 |10 | 4 | 5 | 1 | 3 | 1 | 2 | 1 | 1 | 1    | 303    |
7. There are \((2)(6)(3) = 36\) simulation runs. The number of data sets (i.e., simulation iterations) for each simulation run are 1,000, 2,000, 5,000, or 10,000 depending on the run. These numbers were chosen to make the standard deviations of the estimated percentage of errors (soon to be defined precisely) of the various estimators less than 3 and, in most cases, less than 1. These numbers are shown in Table 6.

8. The default prior distribution on \((\theta, \gamma^2)\) (see Sec. 5.2) is used.

### 7.2 Percentage of Error Results

The results of the simulation are given in Table 7. The column labeled “Coverage/standard deviation” is the percentage (along with its standard deviation) of simulation iterations for which the 90% QMP interval estimate (described in Sec. 6) actually contained the true outgoing quality. The four columns labeled “Percentage of error/standard deviation” contain two results separated by a slash. To define these results precisely for a given run and estimator, let \((q_i, \hat{q}_i), j = 1, \ldots, J,\) denote the true and estimated outgoing quality for iteration \(j\) of the simulation. The percentage of error for iteration \(j\) is defined by

\[
P_j = \left| \frac{q_j - \hat{q}_j}{\bar{q}} \right| (100),
\]

where \(\bar{q}\) is the average of the \(q_i\)’s. The results shown in the table are the mean and standard deviation

\[
\bar{p} = \frac{1}{J} \sum_{j=1}^{J} p_j, \quad s_p = \left[ \frac{1}{J} \sum_{j=1}^{J} (p_j - \bar{p})^2 \right]^{1/2}.
\]

Define the relative efficacy (RE) of a particular estimator for a particular run as

\[
\text{RE} = \left[ \frac{\text{minimum percent error}}{\text{over all four estimators}} \right] + \left[ \frac{\text{percent error for this estimator}}{1} \right].
\]

So, for a particular simulation run, the estimator with the smallest percentage of error has an RE of 1. Table 8 gives a summary of the number of runs for which an estimator’s RE is in various categories. From Table 8, we see that the QMP estimator has the smallest percentage of error in 27 of 36 runs. For five runs, the QMP is a close second. In four runs, QMP did not perform well. These four runs were for process distribution \((f),\) which is an extreme two-point distribution for the \(q_i\)’s. In practice, such a distribution implies two populations, which usually can be segregated. Then QMP could be applied separately to the two populations.

Notice that all of the other estimators have low RE much more often than QMP. The estimator \(\hat{Q}_2\) has low RE for 30 out of 36 runs. It has the minimum percentage of error for only one run. The performances of \(\hat{Q}_1\) and \(\hat{Q}_2\) are erratic.

A closer examination of Table 7 leads to the following conclusions:

1. For QMP, \(\text{RE} \geq .8\) for all unimodal process distributions.
2. For \(\hat{Q}_1, \text{RE} \geq .8\) for most runs with process variance equal to 0.
3. \(\hat{Q}_2\) behaves erratically for all process distributions.
4. For \(\hat{Q}_2, \text{RE} = 1\) for only one run, which has extreme bimodality and 50 lots.

### 7.3 Coverage Results

Almost all runs in which the coverage is significantly different from 90% (i.e., less than 86%) fall into one of three categories:

1. Process distribution \((f)\)
2. Process distribution \((d)\)
3. Lot/sample scenario \((a)\) with a window size of 10.

The only exception is run 15, where the coverage is 84%.

For category 1, the process distribution \((f)\) is a severe violation of the QMP assumption of a gamma process distribution. For category 2, the mean and variance of the gamma process distribution \((d)\) is \((\theta, \gamma^2) = (2, 2),\) which is in the domain outskirts of the...
Table 7. Simulation Results

<table>
<thead>
<tr>
<th>Run no.</th>
<th>Process dist.</th>
<th>Lot/ sample</th>
<th>Window size</th>
<th>Coverage/ standard deviation</th>
<th>Percentage of error/ standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \hat{Q}_1 )</td>
<td>( \hat{Q}_2 )</td>
</tr>
<tr>
<td>1</td>
<td>a</td>
<td>a</td>
<td>10</td>
<td>82/4</td>
<td>70/6</td>
</tr>
<tr>
<td>2</td>
<td>a</td>
<td>a</td>
<td>50</td>
<td>89/1.0</td>
<td>31/8</td>
</tr>
<tr>
<td>3</td>
<td>a</td>
<td>b</td>
<td>10</td>
<td>90/4</td>
<td>31/3</td>
</tr>
<tr>
<td>4</td>
<td>a</td>
<td>b</td>
<td>10</td>
<td>89/1.0</td>
<td>13/3</td>
</tr>
<tr>
<td>5</td>
<td>a</td>
<td>c</td>
<td>10</td>
<td>91/6</td>
<td>16/3</td>
</tr>
<tr>
<td>6</td>
<td>a</td>
<td>c</td>
<td>50</td>
<td>89/1.0</td>
<td>7/2</td>
</tr>
<tr>
<td>7</td>
<td>b</td>
<td>a</td>
<td>10</td>
<td>85/4</td>
<td>57/4</td>
</tr>
<tr>
<td>8</td>
<td>b</td>
<td>a</td>
<td>50</td>
<td>88/1.0</td>
<td>26/6</td>
</tr>
<tr>
<td>9</td>
<td>b</td>
<td>b</td>
<td>10</td>
<td>90/4</td>
<td>25/3</td>
</tr>
<tr>
<td>10</td>
<td>b</td>
<td>b</td>
<td>50</td>
<td>90/1.0</td>
<td>11/3</td>
</tr>
<tr>
<td>11</td>
<td>b</td>
<td>c</td>
<td>10</td>
<td>92/6</td>
<td>12/2</td>
</tr>
<tr>
<td>12</td>
<td>b</td>
<td>c</td>
<td>50</td>
<td>91/1.0</td>
<td>5/10</td>
</tr>
<tr>
<td>13</td>
<td>c</td>
<td>a</td>
<td>10</td>
<td>81/4</td>
<td>88/7</td>
</tr>
<tr>
<td>14</td>
<td>c</td>
<td>a</td>
<td>50</td>
<td>87/1.1</td>
<td>41/1.1</td>
</tr>
<tr>
<td>15</td>
<td>c</td>
<td>b</td>
<td>10</td>
<td>84/5</td>
<td>60/7</td>
</tr>
<tr>
<td>16</td>
<td>c</td>
<td>b</td>
<td>50</td>
<td>86/1.1</td>
<td>39/8</td>
</tr>
<tr>
<td>17</td>
<td>c</td>
<td>c</td>
<td>10</td>
<td>88/7</td>
<td>42/8</td>
</tr>
<tr>
<td>18</td>
<td>c</td>
<td>c</td>
<td>50</td>
<td>91/9</td>
<td>32/6</td>
</tr>
<tr>
<td>19</td>
<td>d</td>
<td>a</td>
<td>10</td>
<td>83/4</td>
<td>64/5</td>
</tr>
<tr>
<td>20</td>
<td>d</td>
<td>a</td>
<td>50</td>
<td>86/1.1</td>
<td>32/7</td>
</tr>
<tr>
<td>21</td>
<td>d</td>
<td>b</td>
<td>10</td>
<td>80/6</td>
<td>56/6</td>
</tr>
<tr>
<td>22</td>
<td>d</td>
<td>b</td>
<td>50</td>
<td>78/1.2</td>
<td>44/8</td>
</tr>
<tr>
<td>23</td>
<td>d</td>
<td>c</td>
<td>10</td>
<td>83/8</td>
<td>55/9</td>
</tr>
<tr>
<td>24</td>
<td>d</td>
<td>c</td>
<td>50</td>
<td>87/1.1</td>
<td>52/6</td>
</tr>
<tr>
<td>25</td>
<td>e</td>
<td>a</td>
<td>10</td>
<td>83/4</td>
<td>65/5</td>
</tr>
<tr>
<td>26</td>
<td>e</td>
<td>a</td>
<td>50</td>
<td>88/1.0</td>
<td>29/7</td>
</tr>
<tr>
<td>27</td>
<td>e</td>
<td>b</td>
<td>10</td>
<td>90/4</td>
<td>31/4</td>
</tr>
<tr>
<td>28</td>
<td>e</td>
<td>b</td>
<td>50</td>
<td>91/1.0</td>
<td>14/4</td>
</tr>
<tr>
<td>29</td>
<td>e</td>
<td>c</td>
<td>10</td>
<td>90/7</td>
<td>17/3</td>
</tr>
<tr>
<td>30</td>
<td>e</td>
<td>c</td>
<td>50</td>
<td>88/1.0</td>
<td>8/2</td>
</tr>
<tr>
<td>31</td>
<td>f</td>
<td>a</td>
<td>10</td>
<td>80/4</td>
<td>72/6</td>
</tr>
<tr>
<td>32</td>
<td>f</td>
<td>a</td>
<td>50</td>
<td>82/1.2</td>
<td>36/9</td>
</tr>
<tr>
<td>33</td>
<td>f</td>
<td>b</td>
<td>10</td>
<td>56/7</td>
<td>111/1.1</td>
</tr>
<tr>
<td>34</td>
<td>f</td>
<td>b</td>
<td>50</td>
<td>41/1.6</td>
<td>105/1.2</td>
</tr>
<tr>
<td>35</td>
<td>f</td>
<td>c</td>
<td>10</td>
<td>51/1.1</td>
<td>215/2.1</td>
</tr>
<tr>
<td>36</td>
<td>f</td>
<td>c</td>
<td>50</td>
<td>37/1.5</td>
<td>215/13</td>
</tr>
</tbody>
</table>

Table 8. Summary of Relative Efficacy

<table>
<thead>
<tr>
<th>Relative efficacy</th>
<th>( \hat{Q}_1 )</th>
<th>( \hat{Q}_2 )</th>
<th>( \hat{Q}_3 )</th>
<th>QMP</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE = 1</td>
<td>6</td>
<td>8</td>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td>0.8 ( \leq ) RE &lt; 1</td>
<td>17</td>
<td>3</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0.5 ( \leq ) RE &lt; 0.8</td>
<td>6</td>
<td>13</td>
<td>25</td>
<td>3</td>
</tr>
<tr>
<td>RE &lt; 0.5</td>
<td>7</td>
<td>12</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 9. Simulation of Example 2b With a Gamma Process Distribution

<table>
<thead>
<tr>
<th>Percentage of error/standard deviation</th>
<th>Coverage/ standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{Q}_1 )</td>
<td>( \hat{Q}_2 )</td>
</tr>
<tr>
<td>-----------------</td>
<td>---------------</td>
</tr>
<tr>
<td>484.19/1.06</td>
<td>100.00/1.17</td>
</tr>
</tbody>
</table>

8. SIMULATION OF EXAMPLE 2b

To get a feel for how well QMP really would work on a data set such as Example 2b, we designed a special simulation run to produce such data. A QMP analysis of Example 2b (with the diffuse prior) yielded an estimated process average and variance of 5 and 191 on an index scale, so we selected a gamma process distribution with mean = 5 and variance = 200 for our simulation.

The results of 10,000 iterations are shown in Table 9. For this simulation, QMP and the nonparametric empirical Bayes method of Zaslavsky (1988) are tied on a data set such as Example 2b, we designed a
marginal distribution of \( x_i \), which is required by the Zaslavsky estimator. Note that \( \hat{O}_Q \) and \( \hat{O}_Q^r \) both perform badly.

To check sensitivity to the gamma assumption, we did a similar simulation in which the process distribution was a mixture of exponentials with overall mean and variance approximately equal to 5 and 200. The mixture was 90% exponential with mean 2 and 10% exponential with mean 33. The surprising results are shown in Table 10. QMP does not work well.

The reason is subtle. Let \( \pi_G \) and \( \pi_M \) denote the gamma and mixture-of-exponentials process distributions with the same mean and variance of 5 and 200. It turns out that \( E \{ \theta \mid x_i = 0, \pi_G \} \) and \( E \{ \theta \mid x_i = 0, \pi_M \} \) are very different, even though \( \pi_G \) and \( \pi_M \) have the same mean and variance. This is because the distributions have different shapes for those small values of \( \theta \) that are consistent with the observation \( x_i = 0 \). And with 303 lots, the nonparametric estimate of \( \pi_M \) is very good for these small values of \( \theta \).

This suggests a seminonparametric empirical Bayes approach. Equation (6) shows how the problem depends on estimating the marginal probability mass function, \( f(x) = \Pr(x = x \mid \pi) \). This could be estimated with a kernel-type smoothing of the empirical probability mass function \( \hat{f}(x) = \frac{T_x}{K} \). The kernel for \( x = c + 1 \) should not go into the tails of \( \hat{f}(x) \). This kind of approach was discussed by Martz (1975).

9. CONCLUSION

The QMP estimator of outgoing quality has many advantages, but it should be used with some care. Some advantages are that (a) it works when the lot sizes, sample sizes, and acceptance numbers vary within the reporting window; (b) it works when the acceptance sampling rules are violated; (c) it provides an interval estimator; (d) it has Bayes optimality for the model assumed; and (e) it is somewhat robust against deviation from assumptions as described in Section 7. The following precautions should be noted in any application, however:

1. The results can be sensitive to the prior on the hyperparameters (process average, process variance). Past data across products should be used to construct this prior in any system application.

2. Avoid severe bimodality in the process distribution. For example, separate initial manufactured and reworked lots into two populations and apply QMP separately.

3. Pay attention to outliers. Outlier rejection or deflation techniques should be used where appropriate.

4. When there are hundreds of lots in the reporting window, test for fit of the gamma process-distribution assumption. If the fit is poor in the sense described in Section 8, then use Zaslavsky's (1988) nonparametric approach or an alternative parametric approach.

[Received July 1987. Revised June 1989.]

REFERENCES


